Inverse iteration for p-ground states

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Abstract

We adapt the inverse iteration method for symmetric matrices to some nonlinear PDE eigenvalue problems. In particular, for $p \in (1,\infty)$ and a given domain $\Omega \subset \mathbb{R}^n$, we analyze a scheme that allows us to approximate the smallest value the ratio $\int_\Omega |D\psi|^p dx / \int_\Omega |\psi|^p dx$ can assume for functions ψ that vanish on $\partial\Omega$. The scheme in question also provides a natural way to approximate minimizing ψ. Our analysis also extends in the limit as $p \to \infty$ and thereby fashions a new approximation method for ground states of the infinity Laplacian.

1 Introduction

In this paper, we will use a generalization of the inverse iteration method for symmetric matrices to estimate solutions of some nonlinear PDE eigenvalue problems. The first problem we consider is as follows. For $p \in (1,\infty)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, we define

$$\lambda_p := \inf\left\{\frac{\int_\Omega |D\psi|^p dx}{\int_\Omega |\psi|^p dx} : \psi \in W_0^{1,p}(\Omega),\ \psi \not\equiv 0\right\}. \tag{1.1}$$

Here $W_0^{1,p}(\Omega)$ is the closure of the smooth, compactly supported functions $\phi : \Omega \to \mathbb{R}$ in the norm $\left(\int_\Omega |D\phi|^p dx\right)^{1/p}$; we refer readers to the sources [4, 9] for information on Sobolev spaces and their applications to PDE . It is evident that $1/\lambda_p$ is the smallest constant C for which the *Poincaré inequality*

$$\int_\Omega |\psi|^p dx \le C \int_\Omega |D\psi|^p dx, \quad \psi \in W_0^{1,p}(\Omega)$$

holds.

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The constant λ_p is also a type of eigenvalue. Indeed, minimizers in (1.1) are called *p-ground states* and satisfy the PDE

$$\begin{cases} -\Delta_p u = \lambda_p |u|^{p-2}u, & x \in \Omega, \\ \quad u = 0, & x \in \partial\Omega. \end{cases}$$

Here, the operator $\Delta_p \psi := \mathrm{div}(|D\psi|^{p-2}D\psi)$ is known as the p-Laplacian. It has been established that p-ground states exist and that any two are multiples of one another, see [8, 11]. Consequently, λ_p is said to be *simple*.

We will use the following iteration scheme to approximate λ_p and p-ground states. Let $u_0 \in L^p(\Omega)$, and consider the family of PDE

$$\begin{cases} -\Delta_p u_k = |u_{k-1}|^{p-2}u_{k-1}, & x \in \Omega \\ \quad u_k = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

for $k \in \mathbb{N}$. It can be verified without too much difficulty that for a given u_0, there is a unique weak solution sequence $(u_k)_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ of (1.2). That is, there is a unique sequence $(u_k)_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |Du_k|^{p-2}Du_k \cdot D\phi dx = \int_\Omega |u_{k-1}|^{p-2}u_{k-1}\phi dx$$

for each $\phi \in W_0^{1,p}(\Omega)$ and $k \in \mathbb{N}$. In fact, once $u_{k-1} \in L^p(\Omega)$ is known, u_k can be obtained by minimizing the functional

$$W_0^{1,p}(\Omega) \ni v \mapsto \int_\Omega \left(\frac{1}{p}|Dv|^p - |u_{k-1}|^{p-2}u_{k-1}v \right) dx.$$

As this functional is strictly convex and coercive, the existence of a unique minimizer follows from the "direct method" of the calculus of variations.

The following theorem details how the scheme (1.2) is related to λ_p and p-ground states.

Theorem 1.1. *Assume $u_0 \in L^p(\Omega)$ and define*

$$\mu_p := \lambda_p^{\frac{1}{p-1}}.$$

Then the limit

$$\psi := \lim_{k\to\infty} \mu_p^k u_k$$

exists in $W_0^{1,p}(\Omega)$. If $\psi \not\equiv 0$, then ψ is a p-ground state and

$$\lambda_p = \lim_{k\to\infty} \frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx}. \tag{1.3}$$

Remark 1.2. It may not be obvious how to verify that the limiting function ψ is not identically zero. However, if for instance $u_0 > 0$ in $\overline{\Omega}$ or if $u_0 \geq 0$ and Ω is regular enough in order to have a Hopf's lemma (for instance $C^{1,\alpha}$, cf. [10]), then it is straightforward to verify that ψ is indeed non-zero.

The iteration scheme (1.2) was introduced by R. Biezuner, G. Ercole, and E. Martins in [1] who conjectured the limit

$$\lambda_p = \lim_{k\to\infty} \left(\frac{\int_\Omega |u_{k-1}|^p dx}{\int_\Omega |u_k|^p dx}\right)^{1-1/p}. \tag{1.4}$$

We prove this limit holds under the hypotheses of Theorem 1.1; see Corollary 2.3. We also show that the sequences

$$\left(\frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx}\right)_{k\in\mathbb{N}} \quad \text{and} \quad \left(\frac{\int_\Omega |u_{k-1}|^p dx}{\int_\Omega |u_k|^p dx}\right)_{k\in\mathbb{N}}$$

are nonincreasing, which we regard as special features of the the iteration (1.2). See Proposition 2.4 below.

Next, we derive an iteration scheme in the limit as $p \to \infty$. Our motivation was the seminal work of P. Juutinen, P. Lindqvist, and J. Manfredi [6], where it was proven that $\lim_{p\to\infty} \lambda_p^{1/p}$ exists and equals

$$\lambda_\infty := \inf\left\{\frac{|D\psi|_{L^\infty(\Omega)}}{|\psi|_{L^\infty(\Omega)}} : \psi \in W_0^{1,\infty}(\Omega),\ \psi \not\equiv 0\right\} = \left(\sup\{r : B_r(x) \subset \Omega \text{ for some } x \in \Omega\}\right)^{-1}.$$

Here $W_0^{1,\infty}(\Omega)$ is the space of Lipschitz continuous functions $\psi : \overline{\Omega} \to \mathbb{R}$ that satisfy $\psi|_{\partial\Omega} = 0$. Furthermore, these authors also showed that there is a sequence $(u_{p_j})_{j\in\mathbb{N}}$ of p-ground states that converge uniformly to a viscosity solution $w \in W_0^{1,\infty}(\Omega)$ of the PDE

$$0 = \begin{cases} \min\{-\Delta_\infty w, |Dw| - \lambda_\infty w\}, & w > 0, \\ -\Delta_\infty w, & w = 0, \\ \max\{-\Delta_\infty w, -|Dw| - \lambda_\infty w\}, & w < 0. \end{cases} \tag{1.5}$$

Here $\Delta_\infty \psi := D^2\psi D\psi \cdot D\psi$ is the infinity Laplacian and nontrivial solutions of (1.5) having constant sign, are called *∞-ground states*.

Passing to the limit as $p \to \infty$ in (1.2), we are able to conclude the subsequent result. The novelty in the theorem below is that (1.6) presents a new mechanism for generating ∞-ground states.

Theorem 1.3. *Assume $u_0 \in C(\overline{\Omega})$ and denote $(u_{k,p})_{k\in\mathbb{N}}$ as the solution sequence of* (1.2).
(i) There is a sequence $(p_j)_{j\in\mathbb{N}}$ increasing to ∞ and $(v_k)_{k\in\mathbb{N}} \subset W_0^{1,\infty}(\Omega)$ such that $u_{k,p_j} \to v_k$ uniformly on $\overline{\Omega}$ as $j \to \infty$ for each $k \in \mathbb{N}$. Moreover, v_k is a viscosity solution of the PDE

$$0 = \begin{cases} \min\{-\Delta_\infty v_k, |Dv_k| - v_{k-1}\}, & v_{k-1} > 0 \\ -\Delta_\infty v_k, & v_{k-1} = 0 \\ \max\{-\Delta_\infty v_k, -|Dv_k| - v_{k-1}\}, & v_{k-1} < 0 \end{cases} \tag{1.6}$$

for each $k \in \mathbb{N}$. (Here $v_0 := u_0$.)
(ii) The limit $L := \lim_{k\to\infty} \lambda_\infty^k |Dv_k|_{L^\infty(\Omega)}$ exists. If $L > 0$,

$$\lambda_\infty = \lim_{k\to\infty} \frac{|Dv_k|_{L^\infty(\Omega)}}{|v_k|_{L^\infty(\Omega)}}.$$

and any uniformly convergent subsequence of $(\lambda_\infty^k v_k)_{k\in\mathbb{N}}$ converges to a solution of (1.5).

Remark 1.4. Obviously, if $u_0 \geq 0$ and $L > 0$, then any uniformly convergent subsequence of $(\lambda_\infty^k v_k)_{k\in\mathbb{N}}$ converges to an ∞-ground state.

We would especially like to thank Richard Tapia. After learning about our previous work [5] which employed a doubly nonlinear flow to approximate λ_p and p-ground states, Professor Tapia suggested that it may be possible to use inverse iteration to obtain similar results. As noted above, the authors R. Biezuner, G. Ercole, and E. Martins were the first to make this observation in [1]. Nevertheless, we believe this paper adds significantly to [1] and makes clear the connection between inverse iteration and p-ground states.

2 Convergence of the scheme

Before proving Theorem 1.1, we will first make an observation which illuminates how μ_p enters the statement of the theorem. In particular, we will argue that $(\mu_p^k u_k)_{k\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and $\left(\mu_p^k |Du_k|_{L^p(\Omega)}\right)_{k\in\mathbb{N}}$ is a nonincreasing sequence of real numbers.

Lemma 2.1. *For each $k \in \mathbb{N}$,*

$$\mu_p^p \int_\Omega |Du_{k+1}|^p dx \leq \int_\Omega |Du_k|^p dx.$$

Proof. Assume $\int_\Omega |Du_{k+1}|^p dx \neq 0$. We employ Hölder's inequality and the Poincaré inequality to find

$$\begin{aligned}
\int_\Omega |Du_{k+1}|^p dx &= \int_\Omega |Du_{k+1}|^{p-2} Du_{k+1} Du_{k+1} dx \\
&= \int_\Omega |u_k|^{p-2} u_k u_{k+1} dx \\
&\leq \left(\int_\Omega |u_k|^p dx\right)^{1-1/p} \left(\int_\Omega |u_{k+1}|^p dx\right)^{1/p} \qquad (2.1)\\
&\leq \left(\frac{1}{\lambda_p}\int_\Omega |Du_k|^p dx\right)^{1-1/p} \left(\frac{1}{\lambda_p}\int_\Omega |Du_{k+1}|^p dx\right)^{1/p} \\
&= \frac{1}{\lambda_p}\left(\int_\Omega |Du_k|^p dx\right)^{1-1/p} \left(\int_\Omega |Du_{k+1}|^p dx\right)^{1/p}.
\end{aligned}$$

Consequently,

$$\int_\Omega |Du_{k+1}|^p dx \le \frac{1}{\lambda_p^{p/(p-1)}} \int_\Omega |Du_k|^p dx$$

which proves the claim. □

Remark 2.2. A minor variation in the proof of Lemma 2.1 gives the estimate

$$\int_\Omega |Du_k|^p dx \le \frac{1}{\mu_p} \int_\Omega |u_{k-1}|^p dx \tag{2.2}$$

for each $k \in \mathbb{N}$. This estimate will be employed in the proof of Theorem 1.3.

Proof of Theorem 1.1. Set $\psi_k := \mu_p^k u_k$ ($k \in \mathbb{N}$) and

$$S := \lim_{k\to\infty} \int_\Omega |D\psi_k|^p dx.$$

Observe that the limit defining S exists by Lemma 2.1. If $S = 0$, the assertion follows. Let us now assume otherwise.

Notice that $(\psi_k)_{k\in\mathbb{N}}$ satisfies the sequence of PDE

$$\begin{cases} -\Delta_p \psi_k = \lambda_p |\psi_{k-1}|^{p-2}\psi_{k-1}, & x \in \Omega, \\ \qquad \psi_k = 0, & x \in \partial\Omega. \end{cases}$$

By Lemma 2.1 and Rellich-Kondrachov compactness, there is $\psi \in W_0^{1,p}(\Omega)$ and a subsequence $(\psi_{k_j})_{j\in\mathbb{N}}$ so that $\psi_{k_j} \to \psi$ in $L^p(\Omega)$ and $D\psi_{k_j} \rightharpoonup D\psi$ in $L^p(\Omega;\mathbb{R}^n)$, as $j \to \infty$. Also note

$$\int_\Omega |D\psi_{k_j}|^p dx = \int_\Omega |D\psi_{k_j}|^{p-2} D\psi_{k_j} \cdot D\psi_{k_j} dx = \lambda_p \int_\Omega |\psi_{k_j-1}|^{p-2}\psi_{k_j-1}\psi_{k_j} dx.$$

Since $\psi_{k_j} \to \psi$ in $L^p(\Omega)$,

$$\limsup_{j\to\infty} \int_\Omega |D\psi_{k_j}|^p dx = \lambda_p \int_\Omega |\psi|^p dx \le \int_\Omega |D\psi|^p dx.$$

And the weak convergence $D\psi_{k_j} \rightharpoonup D\psi$ in $L^p(\Omega;\mathbb{R}^n)$ gives

$$\liminf_{j\to\infty} \int_\Omega |D\psi_{k_j}|^p dx \ge \int_\Omega |D\psi|^p dx.$$

Thus, $\psi_{k_j} \to \psi$ in $W_0^{1,p}(\Omega)$, $S = \int_\Omega |D\psi|^p dx$ and

$$\int_\Omega |D\psi|^p dx = \lambda_p \int_\Omega |\psi|^p dx.$$

As $S > 0$, $\psi \not\equiv 0$ and thus ψ is a p-ground state. Since S is the same for all any subsequential limit, the simplicity of λ_p implies that $\psi_k \to \psi$ in $W_0^{1,p}(\Omega)$ as claimed. Moreover,

$$\lim_{k\to\infty} \frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx} = \lim_{k\to\infty} \frac{\int_\Omega |D\psi_k|^p dx}{\int_\Omega |\psi_k|^p dx} = \frac{\int_\Omega |D\psi|^p dx}{\int_\Omega |\psi|^p dx} = \lambda_p.$$

□

Observe that if u_0 is a p-ground state, then $(\mu_p^{-k}u_0)_{k\in\mathbb{N}}$ is a "separation of variables" solution of (1.2). This is a trivial case of Theorem 1.1. Also note that $\lim_{k\to\infty}\mu_p^k u_k$ could vanish identically. For instance, this occurs when $p=2$ and u_0 is an eigenfunction of the Dirichlet Laplacian corresponding to an eigenvalue different that λ_2. Let us now see how the limit (1.4) follows from Theorem 1.1.

Corollary 2.3. *Assume* $\lim_{k\to\infty}\mu_p^k|Du_k|_{L^p(\Omega)}\not\equiv 0$, *then the limit* (1.4) *holds.*

Proof. Set $\psi_k:=\mu_p^k u_k$. By the previous assertion, $(\psi_k)_{k\in\mathbb{N}}$ converges to a p-ground state in $W_0^{1,p}(\Omega)$. As a result,

$$\lim_{k\to\infty}\frac{\int_\Omega|u_{k-1}|^p dx}{\int_\Omega|u_k|^p dx}=\mu_p^p\lim_{k\to\infty}\frac{\int_\Omega|\psi_{k-1}|^p dx}{\int_\Omega|\psi_k|^p dx}=\lambda_p^{p/(p-1)}.$$

□

We conclude this section by establishing some fundamental properties of the iteration scheme (1.2). The monotonicity (2.3) suggests the iteration scheme improves the Rayleigh quotient $\int_\Omega|D\psi|^p dx/\int_\Omega|\psi|^p dx$ at each step, and the monotonicity (2.4) gives more insight on the limit (1.4).

Proposition 2.4. *Assume* $u_0\in W_0^{1,p}(\Omega)$ *and* $u_0\not\equiv 0$. *Then* $u_k\not\equiv 0$ *for each* $k\in\mathbb{N}$,

$$\frac{\int_\Omega|Du_{k+1}|^p dx}{\int_\Omega|u_{k+1}|^p dx}\le\frac{\int_\Omega|Du_k|^p dx}{\int_\Omega|u_k|^p dx},\tag{2.3}$$

and

$$\frac{\int_\Omega|u_k|^p dx}{\int_\Omega|u_{k+1}|^p dx}\le\frac{\int_\Omega|u_{k-1}|^p dx}{\int_\Omega|u_k|^p dx}\tag{2.4}$$

for each $k\in\mathbb{N}$.

Proof. If $u_0\not\equiv 0$, then $u_1\not\equiv 0$ or (1.2) could not hold when $k=1$. By induction, we may conclude $u_k\not\equiv 0$ for each $k\in\mathbb{N}$.

Now fix $k\in\mathbb{N}$ and observe

$$\begin{aligned}\int_\Omega|u_k|^p dx&=\int_\Omega(|u_k|^{p-2}u_k)u_k dx\\&=\int_\Omega|Du_{k+1}|^{p-2}Du_{k+1}\cdot Du_k dx\\&\le\left(\int_\Omega|Du_{k+1}|^p dx\right)^{1-1/p}\left(\int_\Omega|Du_k|^p dx\right)^{1/p}.\end{aligned}\tag{2.5}$$

Combining the bound (2.1) with (2.5) gives

$$
\begin{aligned}
\frac{\int_\Omega |Du_{k+1}|^p dx}{\int_\Omega |u_{k+1}|^p dx} &\le \frac{\left(\int_\Omega |u_k|^p dx\right)^{1-1/p}\left(\int_\Omega |u_{k+1}|^p dx\right)^{1/p}}{\int_\Omega |u_{k+1}|^p dx}\\
&= \frac{\int_\Omega |u_k|^p dx}{\left(\int_\Omega |u_{k+1}|^p dx\right)^{1-1/p}\left(\int_\Omega |u_k|^p dx\right)^{1/p}}\\
&\le \frac{\left(\int_\Omega |Du_{k+1}|^p dx\right)^{1-1/p}\left(\int_\Omega |Du_k|^p dx\right)^{1/p}}{\left(\int_\Omega |u_{k+1}|^p dx\right)^{1-1/p}\left(\int_\Omega |u_k|^p dx\right)^{1/p}}\\
&= \left(\frac{\int_\Omega |Du_{k+1}|^p dx}{\int_\Omega |u_{k+1}|^p dx}\right)^{1-1/p}\left(\frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx}\right)^{1/p},
\end{aligned}
$$

which verifies (2.3).

As for (2.4), we employ (2.5), (2.3) and (2.1) to find

$$
\begin{aligned}
\frac{\int_\Omega |u_k|^p dx}{\int_\Omega |u_{k+1}|^p dx} &\le \frac{\left(\int_\Omega |Du_{k+1}|^p dx\right)^{1-1/p}\left(\int_\Omega |Du_k|^p dx\right)^{1/p}}{\int_\Omega |u_{k+1}|^p dx}\\
&\le \left[\int_\Omega |u_{k+1}|^p dx\; \frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx}\right]^{1-1/p} \frac{\left(\int_\Omega |Du_k|^p dx\right)^{1/p}}{\int_\Omega |u_{k+1}|^p dx}\\
&= \frac{\int_\Omega |Du_k|^p dx}{\left(\int_\Omega |u_{k+1}|^p dx\right)^{1/p}\left(\int_\Omega |u_k|^p dx\right)^{1-1/p}}\\
&\le \frac{\left(\int_\Omega |u_k|^p dx\right)^{1/p}\left(\int_\Omega |u_{k-1}|^p dx\right)^{1-1/p}}{\left(\int_\Omega |u_{k+1}|^p dx\right)^{1/p}\left(\int_\Omega |u_k|^p dx\right)^{1-1/p}}\\
&= \left(\frac{\int_\Omega |u_k|^p dx}{\int_\Omega |u_{k+1}|^p dx}\right)^{1/p}\left(\frac{\int_\Omega |u_{k-1}|^p dx}{\int_\Omega |u_k|^p dx}\right)^{1-1/p}.
\end{aligned}
$$

$\square$

Remark 2.5. If $u_0 \not\equiv 0$, the sequences

$$
\left(\frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx}\right)_{k\in\mathbb{N}} \quad \text{and} \quad \left(\frac{\int_\Omega |u_{k-1}|^p dx}{\int_\Omega |u_k|^p dx}\right)_{k\in\mathbb{N}}
$$

are bounded below by λ_p and $\lambda_p^{p/(p-1)}$, respectively; see Proposition 2.8 of [1]. In view of the monotonicity (2.3) and (2.4), both of these sequences are convergent. However, the limits (1.3) and (1.4) may not hold if $\lim_{k\to\infty}\mu_p^k u_k \equiv 0$. For example, these limits fail if $p = 2$ and u_0 is an eigenfunction of the Dirichlet Laplacian that corresponds to an eigenvalue not equal to λ_2.

3 The large p limit

This section is dedicated to a proof of Theorem 1.3, which characterizes the large p limit of the solutions of the iteration scheme (1.2). We begin with an important observation regarding weak solution sequences $(u_k)_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega)$ of (1.2) when $u_0 \in C(\overline{\Omega})$

Lemma 3.1. *Suppose $u_0 \in C(\overline{\Omega})$, and let $(u_k)_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega)$ denote the associated solution sequence of (1.2). Then for each $k \in \mathbb{N}$, there is $\alpha_k \in (0,1)$ such that*

$$u_k \in C^{1,\alpha_k}_{loc}(\Omega) \cap L^\infty(\Omega).$$

Proof. It suffices to verify the claim for $k = 1$; the case $k \geq 2$ then follows from induction. Recall that (1.2) implies $u_1 \in W^{1,p}_0(\Omega)$ is a weak solution of solution of

$$\begin{cases} -\Delta_p u_1 = |u_0|^{p-2}u_0, & x \in \Omega, \\ \quad u_1 = 0, & x \in \partial\Omega. \end{cases}$$

We will use a weak comparison principle argument to bound u_1 from above and then from below. The regularity theory developed by E. DiBenedetto in [3] would then imply the existence of an $\alpha_1 \in (0,1)$ such that $u_1 \in C^{1,\alpha_1}_{\text{loc}}(\Omega)$.

To this end, we fix any $y \notin \overline{\Omega}$ and define

$$w(x) := \frac{1}{qn^{\frac{1}{p-1}}}|x-y|^q, \quad x \in \overline{\Omega}.$$

Here $q = p/(p-1)$ is the Hölder exponent dual to p. Direct computation has $\Delta_p w(x) = 1$ for each $x \in \Omega$. It is also routine to verify that

$$v := |u_0|_{L^\infty(\Omega)}\left(|w|_{L^\infty(\Omega)} - w\right)$$

satisfies

$$-\Delta_p v \geq |u_0|^{p-2}u_0, \quad x \in \Omega.$$

Since $v|_{\partial\Omega} \geq 0 = u_1|_{\partial\Omega}$, a standard weak comparison argument implies $u_1 \leq v$ in Ω. In particular,

$$u_1 \leq |w|_{L^\infty(\Omega)}|u_0|_{L^\infty(\Omega)}, \quad x \in \Omega.$$

We can argue similarly to bound u from below and derive

$$u_1 \geq -|w|_{L^\infty(\Omega)}|u_0|_{L^\infty(\Omega)}, \quad x \in \Omega.$$

□

We have just established that the solution sequence $(u_k)_{k\in\mathbb{N}}$ of the inverse iteration scheme is continuous, provided that u_0 is continuous. Virtually the same argument given by P. Juutinen, P. Lindqvist and J. Manfredi in the proof of Theorem 2.5 of [7] implies that each

u_k is additionally a viscosity solution of (1.2). That is, each solution sequence $(u_k)_{k\in\mathbb{N}} \subset C(\overline{\Omega})$ of (1.2) with $p \geq 2$ has the following property. For each $k \in \mathbb{N}$,

$$-\Delta_p \phi(x_0) \leq |u_{k-1}(x_0)|^{p-2} u_{k-1}(x_0)$$

whenever $\phi \in C^2(\Omega)$ and $u_k - \phi$ has a local maximum at $x_0 \in \Omega$, and

$$-\Delta_p \phi(x_0) \geq |u_{k-1}(x_0)|^{p-2} u_{k-1}(x_0)$$

whenever $\phi \in C^2(\Omega)$ and $u_k - \phi$ has a local minimum at $x_0 \in \Omega$. We refer interested readers to the "user's guide to viscosity solutions" [2] for more information on viscosity solutions of elliptic PDE, and we are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 part (i). Employing Lemma 2.1 and inequality (2.2) for $k = 1$ gives

$$|Du_{k,p}|_{L^p(\Omega)} \leq \frac{1}{\mu_p^{k-1}} |Du_{1,p}|_{L^p(\Omega)} \leq \frac{1}{\mu_p^{k-1+1/p}} |u_0|_{L^p(\Omega)} \leq \frac{|\Omega|^{1/p}}{\mu_p^{k-1+1/p}} |u_0|_{L^\infty(\Omega)}.$$

Assume $p_0 > n$. For $p > p_0$, we can use the above inequality with Hölder's inequality to get

$$|Du_{k,p}|_{L^{p_0}(\Omega)} \leq |\Omega|^{\frac{1}{p_0} - \frac{1}{p}} |Du_{k,p}|_{L^p(\Omega)} \leq \frac{|\Omega|^{1/p_0}}{\mu_p^{k-1+1/p}} |u_0|_{L^\infty(\Omega)}.$$

By Morrey's inequality and $\lim_{p\to\infty} \mu_p = \lambda_\infty$,

$$(u_{k,p})_{p>p_0} \subset C^{1-n/p_0}(\Omega)$$

is bounded for each $k \in \mathbb{N}$. Therefore, the Arzelà-Ascoli Theorem and a typical diagonalization argument implies there is a sequence $(v_k)_{k\in\mathbb{N}} \subset C^{1-n/p_0}(\Omega)$ and a sequence of positive numbers $(p_j)_{j\in\mathbb{N}}$ that are increasing and unbounded such that

$$v_k = \lim_{j\to\infty} u_{k,p_j}$$

in $C^{1-n/p_0}(\Omega)$ for each $k \in \mathbb{N}$.

Now let $p > r$, and employ Hölder's inequality and (2.2) to get

$$\begin{aligned}
\left(\frac{1}{|\Omega|}\int_\Omega |Du_{k,p}|^r dx\right)^{1/r} &\leq \left(\frac{1}{|\Omega|}\int_\Omega |Du_{k,p}|^p dx\right)^{1/p} \\
&\leq \left(\frac{1}{|\Omega|}\frac{1}{\mu_p}\int_\Omega |u_{k-1,p}|^p dx\right)^{1/p} \\
&\leq \frac{1}{\mu_p^{1/p}} |u_{k-1,p}|_{L^\infty(\Omega)}.
\end{aligned}$$

The sequence $(u_{k,p_j})_{j\ge j_r}$ is then bounded in $W_0^{1,r}(\Omega)$ for some $j_r\in\mathbb{N}$ large enough and thus converges to v_k weakly. Therefore, we can substitute $p=p_j$ above and send $j\to\infty$ to arrive at

$$\left(\frac{1}{|\Omega|}\int_\Omega |Dv_k|^r dx\right)^{1/r}\le |v_{k-1}|_{L^\infty(\Omega)}.$$

for each $k\in\mathbb{N}$. And after sending $r\to\infty$,

$$|Dv_k|_{L^\infty(\Omega)}\le |v_{k-1}|_{L^\infty(\Omega)}. \tag{3.1}$$

In particular, we have verified that $(v_k)_{k\in\mathbb{N}}\subset W_0^{1,\infty}(\Omega)$.

We will now verify that v_k are viscosity solutions of the iteration scheme (1.6). By induction, it suffices to prove this for $k=1$. Assume $\phi\in C^2(\Omega)$ and $v_1-\phi$ has a local maximum at $x_0\in\Omega$. We aim to show,

$$0\ge\begin{cases}\min\{-\Delta_\infty\phi(x_0), |D\phi(x_0)|-u_0(x_0)\}, & u_0(x_0)>0,\\ -\Delta_\infty\phi(x_0), & u_0(x_0)=0,\\ \max\{-\Delta_\infty\phi(x_0), -|D\phi(x_0)|-u_0(x_0)\}, & u_0(x_0)<0.\end{cases} \tag{3.2}$$

After adding $x\mapsto\frac{\rho}{2}|x-x_0|^2$ to ϕ and later sending $\rho\to 0^+$, we may assume that $v_1-\phi$ has a *strict* local maximum. Since u_{1,p_j} converges to v_1 uniformly on Ω, there is a sequence $(x_j)_{j\in\mathbb{N}}\subset\Omega$ converging to x_0 for which $u_{1,p_j}-\phi$ has a local maximum at x_j. Since u_{1,p_j} is a viscosity solution of (1.2) with $k=1$ and $p=p_j$,

$$-\Delta_{p_j}\phi(x_j)\le |u_0(x_j)|^{p_j-2}u_0(x_j). \tag{3.3}$$

If $u_0(x_0)<0$, then $u_0(x_j)<0$ for all j sufficiently large. By (3.3),

$$-\Delta_{p_j}\phi(x_j)=|D\phi(x_j)|^{p_j-4}\left\{|D\phi(x_j)|^2\Delta\phi(x_j)+(p_j-2)\Delta_\infty\phi(x_j)\right\}<0, \tag{3.4}$$

and thus $|D\phi(x_j)|\ne 0$ all large enough $j\in\mathbb{N}$. Canceling the factor of $|D\phi(x_j)|^{p_j-4}$ in (3.4), dividing by p_j-2 and sending $j\to\infty$ gives $-\Delta_\infty\phi(x_0)\le 0$. Likewise, rearranging (3.3) leads to

$$-\frac{|D\phi(x_j)|^2\Delta\phi(x_j)}{p_j-2}-\Delta_\infty\phi(x_j)\le\frac{1}{p_j-2}\left(\frac{|u_0(x_j)|}{|D\phi(x_j)|}\right)^{p_j-4}u_0(x_j)^3. \tag{3.5}$$

Therefore, it must also be that $-u_0(x_j)\le|D\phi(x_j)|$ for all j large enough. Hence, (3.2) holds when $u_0(x_0)<0$.

Now suppose $u_0(x_0)=0$. If additionally, $|D\phi(x_0)|=0$, then clearly $-\Delta_\infty\phi(x_0)\le 0$. If $|D\phi(x_0)|\ne 0$, we can send $j\to\infty$ in (3.5) to again arrive at $-\Delta_\infty\phi(x_0)\le 0$. Thus, (3.2) holds when $u_0(x_0)=0$.

Finally, let us assume that $u_0(x_0)>0$, and that $|D\phi(x_0)|-u_0(x_0)>0$. Then $|D\phi(x_j)|-u_0(x_j)>0$ for all $j\in\mathbb{N}$ sufficiently large. Passing to the limit in (3.5) again gives $-\Delta_\infty\phi(x_0)\le 0$. In conclusion, (3.2) holds in the case $u_0(x_0)>0$, as well. Therefore, we have verified that v_1 is a viscosity subsolution of (1.6). An argument that shows v_1 is additionally a viscosity supersolution of (1.6) can be made similarly, so we leave the details to the reader. □

Proof of Theorem 1.3 part (ii). In view of (3.1),

$$|Dv_k|_{L^\infty(\Omega)} \le |v_{k-1}|_{L^\infty(\Omega)} \le \frac{1}{\lambda_\infty}|Dv_{k-1}|_{L^\infty(\Omega)}.$$

Therefore, the sequence $(\lambda_\infty^k |Dv_k|_{L^\infty(\Omega)})_{k\in\mathbb{N}}$ is nonincreasing, and the limit

$$L := \lim_{k\to\infty} \lambda_\infty^k |Dv_k|_{L^\infty(\Omega)}$$

exists. The inequality (3.1) also implies

$$|v_k|_{L^\infty(\Omega)} \le \frac{1}{\lambda_\infty}|Dv_k|_{L^\infty(\Omega)} \le \frac{1}{\lambda_\infty}|v_{k-1}|_{L^\infty(\Omega)}.$$

Consequently, $(\lambda_\infty^k |v_k|_{L^\infty(\Omega)})_{k\in\mathbb{N}}$ is nonincreasing and the limit

$$M := \lim_{k\to\infty} \lambda_\infty^k |v_k|_{L^\infty(\Omega)}$$

exists, as well.

Observe $\lambda_\infty^k |Dv_k|_{L^\infty(\Omega)} \le \lambda_\infty \left(\lambda_\infty^{k-1}|v_{k-1}|_{L^\infty(\Omega)}\right)$ so that

$$L \le \lambda_\infty M.$$

Moreover, $\lambda_\infty^k |v_k|_{L^\infty(\Omega)} \le \frac{1}{\lambda_\infty}\lambda_\infty^k |Dv_k|_{L^\infty(\Omega)}$, which implies

$$\lambda_\infty M \le L.$$

Thus, $\lambda_\infty M = L$, and when this quantity is nonzero,

$$\lambda_\infty = \lim_{k\to\infty} \frac{|Dv_k|_{L^\infty(\Omega)}}{|v_k|_{L^\infty(\Omega)}}.$$

Finally, note that that the sequence $(w_k)_{k\in\mathbb{N}} := (\lambda_\infty^k v_k)_{k\in\mathbb{N}} \subset W_0^{1,\infty}(\Omega)$ satisfies the iteration scheme

$$0 = \begin{cases} \min\{-\Delta_\infty w_k, |Dw_k| - \lambda_\infty w_{k-1}\}, & w_{k-1} > 0 \\ -\Delta_\infty w_k, & w_{k-1} = 0 \\ \max\{-\Delta_\infty w_k, -|Dw_k| - \lambda_\infty w_{k-1}\}, & w_{k-1} < 0 \end{cases}$$

in the sense of viscosity solutions. Therefore, if a subsequence of $(\lambda_\infty^k v_k)_{k\in\mathbb{N}}$ converges uniformly on Ω, the stability of viscosity solutions implies that the limit function is necessarily a solution of (1.5). $\square$

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